Darboux transformation for a generalized Hirota-Satsuma coupled Korteweg–de Vries equation

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A Darboux transformation for the generalized Hirota-Satsuma coupled Korteweg–de Vries (KdV) equation is derived with the aid of the gauge transformation between the corresponding 4×4 matrix spectral problems with three potentials, by which some explicit solutions of the generalized Hirota-Satsuma coupled KdV equation are constructed. As a reduction, a Darboux transformation of the complex coupled KdV equation and its explicit solutions are obtained.

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I. INTRODUCTION

In Ref. [1], a hierarchy of coupled nonlinear evolution equations was proposed with the help of a 4×4 matrix spectral problem with three potentials, which contains among its members a generalized Hirota-Satsuma coupled Korteweg–de Vries (KdV) equation

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 3(vw)_x,$$

$$v_t = -v_{xxx} + 3uv_x,$$

$$w_t = -w_{xxx} + 3uw_x. \tag{1}$$

It is shown that the hierarchy of coupled nonlinear evolution equations possesses the generalized Hamiltonian form. Further, a Miura transformation related to the generalized Hirota-Satsuma coupled KdV equation and its reductions are derived, from which some coupled modified KdV equations are obtained. Equation (1) is reduced to the complex coupled KdV equation with $w=v^*$ (complex conjugate),

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 3(|v|^2)_x,$$

$$v_t = -v_{xxx} + 3uv_x, \tag{2}$$

where the potential u is a real function and the potential v is a complex function, and the Hirota-Satsuma coupled KdV equation with $w=v$, which describes an interaction of two long waves with different dispersion relations [2,3]. The generalized Hirota-Satsuma coupled KdV equation has been studied by many authors. Some explicit solutions of Eq. (1), which include soliton solutions, periodic solutions, and other ones, were constructed by various approaches, for example, the tanh-function method [4], Jacobi elliptic function method [5], the algebraic method [6], Adomians decomposition method [7], the homotopy perturbation method [8], and others [9–12].

It is known that Darboux transformation is a powerful tool for solving soliton equations [13–17]. The aim of the present paper is to construct a Darboux transformation for the generalized Hirota-Satsuma coupled KdV equation with the aid of a gauge transformation between the corresponding 4×4 matrix spectral problems with three potentials, by which some explicit solutions of the generalized Hirota-Satsuma coupled KdV Eq. (1) are given. As a reduction, a Darboux transformation of the complex coupled KdV Eq. (2) and its explicit solutions are obtained. A systematic algebraic procedure is given in detail to solve the generalized Hirota-Satsuma coupled KdV equation. The present paper is organized as follows. In Sec. II, we shall construct a Darboux transformation of the generalized Hirota-Satsuma coupled KdV equation resorting to the gauge transformation between 4×4 matrix spectral problems. In Sec. III, a Darboux transformation of the complex coupled KdV Eq. (2) is discussed through the reduction technique. Furthermore, in Sec. IV, some explicit solutions of the generalized Hirota-Satsuma coupled KdV Eq. (1) and the complex coupled KdV Eq. (2) are derived by the application of their Darboux transformations.

II. DARBOUX TRANSFORMATION OF THE GENERALIZED HIROTA-SATSUMA COUPLED KDV EQUATION

In this section, we shall construct a Darboux transformation for the generalized Hirota-Satsuma coupled KdV Eq. (1), which comes from the condition of the compatibility for the 4×4 matrix spectral problem with three potentials

$$\chi_x = U(s,\lambda)\chi \tag{3}$$

and an auxiliary problem

$$\chi_t = V(s,\lambda)\chi, \tag{4}$$

where λ is a constant spectral parameter, $\chi = (\chi_1,\chi_2,\chi_3,\chi_4)^T, s=(u,v,w)$, and

$$U=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u+\lambda & v & 0 & 0 \\ w & u-\lambda & 0 & 0 \end{pmatrix},$$

$$V=\begin{pmatrix} \frac{1}{2}u_x & -v_x & -u+2\lambda & 2v \\ -w_x & \frac{1}{2}u_x & 2w & -u-2\lambda \\ \frac{1}{2}u_{xx}+2wv-u^2+\lambda u+2\lambda^2 & -v_{xx}+uv & -\frac{1}{2}u_x & v_x \\ -w_{xx}+uw & \frac{1}{2}u_{xx}+2wv-u^2-\lambda u+2\lambda^2 & w_x & -\frac{1}{2}u_x \end{pmatrix}.$$

In fact, a direct calculation shows that the zero-curvature equation, $U_t-V_x+[U,V]=0$, implies the generalized Hirota-Satsuma coupled KdV Eq. (1).

To construct a Darboux transformation of the generalized Hirota-Satsuma coupled KdV Eq. (1), we assume that $\phi^{(l)}=(\phi_1^{(l)},\phi_2^{(l)},\phi_3^{(l)},\phi_4^{(l)})^T, 1\leq l\leq 4$, are four linearly independent solutions of Eqs. (3) and (4), and use them to define a fundamental matrix of solutions $\Phi=(\phi^{(1)},\phi^{(2)},\phi^{(3)},\phi^{(4)})$. Now we introduce a gauge transformation for the spectral problems (3) and (4), $\Phi\to\bar{\Phi}$:

$$\bar{\Phi}=T\Phi \tag{5}$$

with

$$T=\begin{pmatrix} a+\lambda & b & d & 0 \\ c & -a+\lambda & 0 & -d \\ a_x+du+\lambda d & dv+b_x & a+d_x+\lambda & b \\ -dw+c_x & -a_x-du+\lambda d & c & -a-d_x+\lambda \end{pmatrix}, \tag{6}$$

where $a,b,c,$ and d are determined later. Multiplying the first row and the second one in the determinant $\det T$ by $-d$ and adding them to the third row and the fourth one, respectively, yield $\det T=\det(\lambda I-\hat{T})$, where $\hat{T}$ is a 4×4 matrix independent of λ. Through tedious calculations, we obtain

$$\det T=\lambda^4+f_1\lambda^2+f_2 \tag{7}$$

with

$$f_1=\sum_{1\leq i<j\leq 4}\begin{vmatrix} \hat{T}_{ii} & \hat{T}_{ij} \\ \hat{T}_{ji} & \hat{T}_{jj}\end{vmatrix}, \quad f_2=\det\hat{T}.$$

Let $\lambda_1,\lambda_2(\lambda_1^2\neq\lambda_2^2)$ be two arbitrary given parameters and be the roots of fourth-order polynomial $\det T$, that is

$$\det T=(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2). \tag{8}$$

It is easy to see that the column vectors $\{\bar{\phi}^{(1)},\bar{\phi}^{(2)},\bar{\phi}^{(3)},\bar{\phi}^{(4)}\}$ of $\bar{\Phi}$ are linearly dependent as $\lambda=\lambda_j$ $(j=1,2)$. Therefore, there exist constants $r_k^{(j)}$ (in which a constant is at least not zero) such that

$$r_0^{(j)}\bar{\phi}^{(1)}(\lambda_j)+r_1^{(j)}\bar{\phi}^{(2)}(\lambda_j)+r_2^{(j)}\bar{\phi}^{(3)}(\lambda_j)+r_3^{(j)}\bar{\phi}^{(4)}(\lambda_j)=0.$$

Substituting every column vector of Eq. (5) into the above equation, a direct calculation gives rise to a linear algebraic system

$$a+\lambda_j+\sigma_1^{(j)}b+\sigma_2^{(j)}d=0,$$

$$c+(-a+\lambda_j)\sigma_1^{(j)}-\sigma_3^{(j)}d=0, \tag{9}$$

and

$$a_x+du+\lambda_j d+\sigma_1^{(j)}(dv+b_x)+\sigma_2^{(j)}(a+d_x+\lambda_j)+\sigma_3^{(j)}b=0,$$

$$(-dw+c_x)+\sigma_1^{(j)}(-a_x-du+\lambda_j d)+\sigma_2^{(j)}c+\sigma_3^{(j)}(-a-d_x+\lambda_j)=0, \tag{10}$$

with $(j=1,2)$

$$\sigma_1^{(j)}=\frac{\phi_2^{(1)}(\lambda_j)-r_1^{(j)}\phi_2^{(2)}(\lambda_j)-r_2^{(j)}\phi_2^{(3)}(\lambda_j)-r_3^{(j)}\phi_2^{(4)}(\lambda_j)}{\phi_1^{(1)}(\lambda_j)-r_1^{(j)}\phi_1^{(2)}(\lambda_j)-r_2^{(j)}\phi_1^{(3)}(\lambda_j)-r_3^{(j)}\phi_1^{(4)}(\lambda_j)},$$

$$\sigma_2^{(j)}=\frac{\phi_3^{(1)}(\lambda_j)-r_1^{(j)}\phi_3^{(2)}(\lambda_j)-r_2^{(j)}\phi_3^{(3)}(\lambda_j)-r_3^{(j)}\phi_3^{(4)}(\lambda_j)}{\phi_1^{(1)}(\lambda_j)-r_1^{(j)}\phi_1^{(2)}(\lambda_j)-r_2^{(j)}\phi_1^{(3)}(\lambda_j)-r_3^{(j)}\phi_1^{(4)}(\lambda_j)},$$

$$\sigma_3^{(j)}=\frac{\phi_4^{(1)}(\lambda_j)-r_1^{(j)}\phi_4^{(2)}(\lambda_j)-r_2^{(j)}\phi_4^{(3)}(\lambda_j)-r_3^{(j)}\phi_4^{(4)}(\lambda_j)}{\phi_1^{(1)}(\lambda_j)-r_1^{(j)}\phi_1^{(2)}(\lambda_j)-r_2^{(j)}\phi_1^{(3)}(\lambda_j)-r_3^{(j)}\phi_1^{(4)}(\lambda_j)}, \tag{11}$$

where $r_0^{(j)}=-1$ is taken for the sake of simplicity. Then Eq. (9) implies Eq. (10). In fact, differentiating the first expression of Eqs. (9) and (11) with respect to x, substituting the latter into the former yields the first expression of Eq. (10) with the help of Eq. (3) through tedious calculations. Similarly, we have the second expression of Eq. (10).

Assume that parameters λ_j and $r_k^{(j)}$ $(j=1,2)$ are suitably chosen such that the determinant of the coefficients for Eq. (9) is nonzero. Then $a,b,c,$ and d are uniquely determined by the linear algebraic system (9).

Let us consider a linear transformation

$$\bar{\chi}=T\chi, \tag{12}$$

from which Eqs. (3) and (4) are transformed into two 4×4 matrix spectral problems of $\bar{\chi}$ in the case $\lambda\neq\lambda_j$ as follows:

$$\bar{\chi}_x=\bar{U}(\bar{s},\lambda)\bar{\chi}, \tag{13}$$

$$\bar{\chi}_t=\bar{V}(\bar{s},\lambda)\bar{\chi}, \tag{14}$$

where $\bar{s}=(\bar{u},\bar{v},\bar{w})$,

$$\bar{U}=(T_x+TU)T^{-1}, \quad \bar{V}=(T_t+TV)T^{-1}. \tag{15}$$

It turns out that $\lambda=\lambda_j$ are removable isolated singularities of $\bar{U}$ and $\bar{V}$ (see below). Therefore, we can define $\bar{U}$ and $\bar{V}$ for all $\lambda\neq 0$ by analytic continuation.

Proposition 1. The matrices $\bar{U}(\bar{s},\lambda)$ and $\bar{V}(\bar{s},\lambda)$ determined by Eq. (15) have the same forms as $U(s,\lambda)$ and $V(s,\lambda)$, respectively, where the transformation formulae from original potentials u,v,w into new ones are given by

$$\bar{u}=u+2d_x,$$

$$\bar{v}=v-2b,$$

$$\bar{w}=w+2c. \tag{16}$$

In fact, let $T^{-1}=T^{*}/\det T$, which is a known formula of the inverse matrix, and

$$(T_x+TU)T^{*}=F=[f_{sl}(\lambda)]_{4\times 4}, \quad (T_t+TV)T^{*}=G$$
$$=[g_{sl}(\lambda)]_{4\times 4}, \tag{17}$$

where T^{*} stands for the adjugate matrix of T, every element of the matrix T^{*} is an algebraic cofactors of matrix T. Using Eq. (11) and the spectral problem (3) with $\lambda=\lambda_j(j=1,2)$, we can obtain the first-order differential equations of $\sigma_l^{(j)}$, $(l=1,2,3)$, with respect to x. A direct calculation shows that $f_{sl}(\lambda)$ and $g_{sl}(\lambda)$ are polynomials of λ. Based on these facts, we can verify the proposition through tedious calculations.

According to proposition 1, the transformation, Eqs. (12) and (16), maps the Lax pair (3) and (4) into another Lax pair of the same type:

$$\bar{\chi}_x=\bar{U}(\bar{s},\lambda)\bar{\chi}, \tag{18}$$

$$\bar{\chi}_t=\bar{V}(\bar{s},\lambda)\bar{\chi}. \tag{19}$$

Therefore, both of the Lax pairs lead to the same generalized Hirota-Satsuma coupled KdV Eq. (1). The transformation (16), $(u,v,w)\rightarrow(\bar{u},\bar{v},\bar{w})$, is called a Darboux transformation of the generalized Hirota-Satsuma coupled KdV Eq. (1).

Proposition 2. Every solution (u,v,w) of the generalized Hirota-Satsuma coupled KdV Eq. (1) is mapped into a solution $(\bar{u},\bar{v},\bar{w})$ of Eq. (1) under the Darboux transformation (16), in which a,b,c,d are uniquely determined by Eq. (9).

For the sake of clarity, here we shall describe briefly the procedure of the construction for the Darboux transformation and explicit solutions of the generalized Hirota-Satsuma coupled KdV Eq. (1):

(i) Choose a known solution of Eq. (1). Usually we choose a trivial solution for simpleness.

(ii) Substitute the known solution into Lax pair (3) and (4) and construct a fundamental matrix of solutions Φ.

(iii) Using Eq. (11) and the fundamental matrix of solutions Φ, we construct $\sigma_l^{(j)}$. Substituting $\sigma_l^{(j)}$ into Eq. (9), then the linear algebraic system (9) uniquely determines functions a,b,c and d. Here we require that parameters λ_j and $r_l^{(j)}$ be suitably chosen such that the determinant of the coefficients for Eq. (9) and the denominators of Eq. (11) are nonzeros.

(iv) Substituting the above determined functions a,b,c,d into Eqs. (6) and (16) yields the Darboux matrix T and a solution of Eq. (1), respectively, by using propositions 1 and 2. This process can be done continually and will usually yield a series of solutions.

III. REDUCTION OF THE DARBOUX TRANSFORMATION

In this section, we will discuss a reduction of the Darboux transformation (16), by which a Darboux transformation of the complex coupled KdV Eq. (2) is given. Equation (9) can be rewritten as

$$a+\sigma_1^{(1)}b+\sigma_2^{(1)}d=-\lambda_1,$$

$$a+\sigma_1^{(2)}b+\sigma_2^{(2)}d=-\lambda_2,$$

$$\sigma_1^{(1)}a-c+\sigma_3^{(1)}d=\lambda_1\sigma_1^{(1)},$$

$$\sigma_1^{(2)}a-c+\sigma_3^{(2)}d=\lambda_2\sigma_1^{(2)}, \tag{20}$$

which imply by the Cramer law that

$$a=\frac{g_1}{g}, \quad b=\frac{g_2}{g}, \quad c=\frac{g_3}{g}, \quad d=\frac{g_4}{g}, \tag{21}$$

where

$$g=\begin{vmatrix} 1 & \sigma_1^{(1)} & 0 & \sigma_2^{(1)} \\ 1 & \sigma_1^{(2)} & 0 & \sigma_2^{(2)} \\ \sigma_1^{(1)} & 0 & -1 & \sigma_3^{(1)} \\ \sigma_1^{(2)} & 0 & -1 & \sigma_3^{(2)} \end{vmatrix},$$

$$g_1=\begin{vmatrix} -\lambda_1 & \sigma_1^{(1)} & 0 & \sigma_2^{(1)} \\ -\lambda_2 & \sigma_1^{(2)} & 0 & \sigma_2^{(2)} \\ \lambda_1\sigma_1^{(1)} & 0 & -1 & \sigma_3^{(1)} \\ \lambda_2\sigma_1^{(2)} & 0 & -1 & \sigma_3^{(2)} \end{vmatrix},$$

$$g_2=\begin{vmatrix} 1 & -\lambda_1 & 0 & \sigma_2^{(1)} \\ 1 & -\lambda_2 & 0 & \sigma_2^{(2)} \\ \sigma_1^{(1)} & \lambda_1\sigma_1^{(1)} & -1 & \sigma_3^{(1)} \\ \sigma_1^{(2)} & \lambda_2\sigma_1^{(2)} & -1 & \sigma_3^{(2)} \end{vmatrix},$$

$$g_3 = \begin{vmatrix} 1 & \sigma_1^{(1)} & -\lambda_1 & \sigma_2^{(1)} \\ 1 & \sigma_1^{(2)} & -\lambda_2 & \sigma_2^{(2)} \\ \sigma_1^{(1)} & 0 & \lambda_1\sigma_1^{(1)} & \sigma_3^{(1)} \\ \sigma_1^{(2)} & 0 & \lambda_2\sigma_1^{(2)} & \sigma_3^{(2)} \end{vmatrix},$$

$$g_4 = \begin{vmatrix} 1 & \sigma_1^{(1)} & 0 & -\lambda_1 \\ 1 & \sigma_1^{(2)} & 0 & -\lambda_2 \\ \sigma_1^{(1)} & 0 & -1 & \lambda_1\sigma_1^{(1)} \\ \sigma_1^{(2)} & 0 & -1 & \lambda_2\sigma_1^{(2)} \end{vmatrix}.$$

Here we assume that the parameters $\lambda_1, \lambda_2, r_1^{(1)}, r_2^{(1)}, r_3^{(1)}, r_1^{(2)}, r_2^{(2)}, r_3^{(2)}$ are suitably chosen so that the determinant of coefficients $g \neq 0$. Substituting Eq. (21) into Eq. (16) we arrive at the explicit form of the Darboux transformation:

$$\bar{u} = u + 2\left(\frac{g_4}{g}\right)_x, \quad \bar{v} = v - 2\frac{g_2}{g}, \quad \bar{w} = w + 2\frac{g_3}{g}. \tag{22}$$

In the following, we shall derive a Darboux transformation of the complex coupled KdV Eq. (2) through the reduction technique. Under the condition $w=v^*$, it is easy to verify that

$$[\phi_2^{(j)*}(-\lambda^*), \phi_1^{(j)*}(-\lambda^*), \phi_4^{(j)*}(-\lambda^*), \phi_3^{(j)*}(-\lambda^*)]^T, (j=1,2),$$

is another solution of Eqs. (3) and (4) if

$$[\phi_1^{(j)}(\lambda), \phi_2^{(j)}(\lambda), \phi_3^{(j)}(\lambda), \phi_4^{(j)}(\lambda)]^T, (j=1,2),$$

is a solution of Eqs. (3) and (4), by which we obtain a fundamental matrix of solutions

$$\begin{pmatrix} \phi_1^{(1)}(\lambda) & \phi_1^{(2)}(\lambda) & \phi_2^{(1)*}(-\lambda^*) & \phi_2^{(2)*}(-\lambda^*) \\ \phi_2^{(1)}(\lambda) & \phi_2^{(2)}(\lambda) & \phi_1^{(1)*}(-\lambda^*) & \phi_1^{(2)*}(-\lambda^*) \\ \phi_3^{(1)}(\lambda) & \phi_3^{(2)}(\lambda) & \phi_4^{(1)*}(-\lambda^*) & \phi_4^{(2)*}(-\lambda^*) \\ \phi_4^{(1)}(\lambda) & \phi_4^{(2)}(\lambda) & \phi_3^{(1)*}(-\lambda^*) & \phi_3^{(2)*}(-\lambda^*) \end{pmatrix}. \tag{23}$$

Then we have the following assertion.

Proposition 3. The restricted condition $w=v^*$ is invariant under the Darboux transformation (22) with the fundamental matrix of solutions (23) and parameters:

$$\lambda_2 = -\lambda_1^*, (\lambda_1 \neq \pm\lambda_2), \quad \frac{r_3^{(1)}}{r_2^{(1)}} = -r_1^{(2)*},$$

$$r_2^{(1)} r_2^{(2)*} = 1, \quad \frac{r_1^{(1)}}{r_2^{(1)}} = -r_3^{(2)*}.$$

This means that the first two formulae in Eq. (22) are a Darboux transformation for the complex coupled KdV Eq. (2) under the above assumption condition.

In fact, we obtain by using Eqs. (11) and (23) and noticing the special choice of parameters that

$$\sigma_1^{(1)}\sigma_1^{(2)*} = 1, \quad \sigma_1^{(1)}\sigma_3^{(2)*} = \sigma_2^{(1)}, \quad \sigma_1^{(2)}\sigma_3^{(1)*} = \sigma_2^{(2)}. \tag{24}$$

With the help of Eq. (24) and $\lambda_2 = -\lambda_1^*$, the result in proposition 3 can be verified after some row and column transformations in the determinants.

IV. EXPLICIT SOLUTIONS

In what follows, we shall apply the Darboux transformations to construct explicit solutions of the generalized Hirota-Satsuma coupled Eq. (1) and the complex coupled KdV Eq. (2).

A. Explicit solutions of the generalized Hirota-Satsuma coupled Eq. (1)

(1) Choose a trivial solution $u=0, v=0, w=0$ of the generalized Hirota-Satsuma coupled Eq. (1). Then Eqs. (3) and (4) are reduced to

$$\begin{cases} \phi_{1,xx} = \lambda\phi_1, & \phi_{1,t} = 2\lambda\phi_{1,x}, \quad \phi_3 = \phi_{1,x}, \\ \phi_{2,xx} = -\lambda\phi_2, & \phi_{2,t} = -2\lambda\phi_{2,x}, \quad \phi_4 = \phi_{2,x}. \end{cases} \tag{25}$$

Equation (25) has a fundamental matrix of solutions with $\lambda = k_0^2$.

$$\Phi = \begin{pmatrix} \exp(A_0) & \exp(-A_0) & 0 & 0 \\ 0 & 0 & \sin B_0 & \cos B_0 \\ k_0\exp(A_0) & -k_0\exp(-A_0) & 0 & 0 \\ 0 & 0 & k_0\cos B_0 & -k_0\sin B_0 \end{pmatrix}, \tag{26}$$

which together with Eq. (11) leads to ($\lambda_1 = k_1^2, \lambda_2 = k_2^2$)

$$\sigma_1^{(j)} = \frac{-r_2^{(j)}\sin B_j - r_3^{(j)}\cos B_j}{\exp(A_j) - r_1^{(j)}\exp(-A_j)},$$

$$\sigma_2^{(j)} = \frac{k_j[\exp(A_j) + r_1^{(j)}\exp(-A_j)]}{\exp(A_j) - r_1^{(j)}\exp(-A_j)},$$

$$\sigma_3^{(j)} = \frac{k_j[r_3^{(j)}\sin B_j - r_2^{(j)}\cos B_j]}{\exp(A_j) - r_1^{(j)}\exp(-A_j)}, \quad j=1,2,$$

where

$$A_i = k_i x + 2k_i^3 t, \quad B_i = k_i x - 2k_i^3 t, \quad i = 0,1,2.$$

According to the Darboux transformation Eq. (16), we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1),

$$\bar{u} = 2\left(\frac{(k_1^2+k_2^2)(\sigma_1^{(1)} - \sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}\right)_x,$$

$$\bar{v} = -2\frac{k_2^2 - k_1^2}{\sigma_1^{(1)} - \sigma_1^{(2)}} - 2\frac{(k_1^2+k_2^2)(\sigma_2^{(2)} - \sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}},$$

$$\bar{w} = 2\frac{(k_1^2 - k_2^2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)} - \sigma_1^{(2)}} + 2\frac{(k_1^2+k_2^2)(\sigma_1^{(1)}\sigma_3^{(2)} - \sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}. \tag{27}$$

(2) Choose a trivial solution $u=1, v=0, w=0$ of the generalized Hirota-Satsuma coupled Eq. (1). Then Eqs. (3) and (4) are written as

$$\begin{cases} \phi_{1,xx} = (1+\lambda)\phi_1, & \phi_{1,t} = (2\lambda-1)\phi_{1,x}, \quad \phi_3 = \phi_{1,x}, \\ \phi_{2,xx} = (1-\lambda)\phi_2, & \phi_{2,t} = -(2\lambda+1)\phi_{2,x}, \quad \phi_4 = \phi_{2,x}. \end{cases} \tag{28}$$

(i) Let $\lambda = k_0(|k_0| < 1)$. Equation (28) has a fundamental matrix of solutions

$$\Phi = \begin{pmatrix} \exp(A_0) & \exp(-A_0) & 0 & 0 \\ 0 & 0 & \exp(B_0) & \exp(-B_0) \\ \sqrt{1+k_0}\exp(A_0) & -\sqrt{1+k_0}\exp(-A_0) & 0 & 0 \\ 0 & 0 & \sqrt{1-k_0}\exp(B_0) & -\sqrt{1-k_0}\exp(-B_0) \end{pmatrix}, \tag{29}$$

which together with Eq. (11) gives rise to $(\lambda_1 = k_1, \lambda_2 = k_2)$,

$$\sigma_1^{(j)} = \frac{-r_2^{(j)}\exp(B_j) - r_3^{(j)}\exp(-B_j)}{\exp(A_j) - r_1^{(j)}\exp(-A_j)},$$

$$\sigma_2^{(j)} = \sqrt{1+k_j}\frac{\exp(A_j) + r_1^{(j)}\exp(-A_j)}{\exp(A_j) - r_1^{(j)}\exp(-A_j)},$$

$$\sigma_3^{(j)} = \sqrt{1-k_j}\frac{-r_2^{(j)}\exp(B_j) + r_3^{(j)}\exp(-B_j)}{\exp(A_j) - r_1^{(j)}\exp(-A_j)}, \quad j = 1,2,$$

with

$$A_i = \sqrt{1+k_i}x + (2k_i - 1)\sqrt{1+k_i}t, \quad B_i = \sqrt{1-k_i}x - (2k_i + 1)\sqrt{1-k_i}t, \quad i = 0,1,2.$$

By employing Darboux transformation (16), we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1),

$$\bar{u} = 1 + 2\left(\frac{(k_1+k_2)(\sigma_1^{(1)} - \sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}\right)_x,$$

$$\bar{v} = -2\frac{k_2 - k_1}{\sigma_1^{(1)} - \sigma_1^{(2)}} - 2\frac{(k_1+k_2)(\sigma_2^{(2)} - \sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}},$$

$$\bar{w} = 2\frac{(k_1 - k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)} - \sigma_1^{(2)}} + 2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)} - \sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}. \tag{30}$$

(ii) Let $\lambda = k_0(k_0 > 1)$. Equation (28) has a fundamental matrix of solutions

$$\Phi = \begin{pmatrix} \sinh(A_0) & \cosh(A_0) & 0 & 0 \\ 0 & 0 & \sin C_0 & \cos C_0 \\ \sqrt{1+k_0}\cosh(A_0) & \sqrt{1+k_0}\sinh(A_0) & 0 & 0 \\ 0 & 0 & \sqrt{k_0-1}\cos C_0 & -\sqrt{k_0-1}\sin C_0 \end{pmatrix}, \tag{31}$$

which together with Eq. (11) gives $[(\lambda_1 = k_1, \lambda_2 = k_2)]$

$$\sigma_1^{(j)} = \frac{-r_2^{(j)}\sin C_j - r_3^{(j)}\cos C_j}{\sinh(A_j) - r_1^{(j)}\cosh(A_j)},$$

$$\sigma_2^{(j)} = \sqrt{1+k_j}\frac{\cosh(A_j) - r_1^{(j)}\sinh(A_j)}{\sinh(A_j) - r_1^{(j)}\cosh(A_j)},$$

$$\sigma_3^{(j)} = \sqrt{k_j - 1}\frac{-r_2^{(j)}\cos C_j + r_3^{(j)}\sin C_j}{\sinh(A_j) - r_1^{(j)}\cosh(A_j)}, \quad j = 1,2,$$

where

$$A_i = \sqrt{1+k_i}x + (2k_i - 1)\sqrt{1+k_i}t, \quad C_i = \sqrt{k_i - 1}x - (2k_i + 1)\sqrt{k_i - 1}t, \quad i = 0,1,2.$$

According to the Darboux transformation (16), we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1),

$$\bar{u} = 1 + 2\left(\frac{(k_1+k_2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}\right)_x,$$

$$\bar{v} = -2\frac{k_2-k_1}{\sigma_1^{(1)}-\sigma_1^{(2)}} - 2\frac{(k_1+k_2)(\sigma_2^{(2)}-\sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}},$$

$$\bar{w} = 2\frac{(k_1-k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)}-\sigma_1^{(2)}} + 2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)}-\sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}. \quad (32)$$

(iii) Let $\lambda = k_0 (k_0 < -1)$. We obtain a fundamental matrix of solutions of Eq. (28),

$$\Phi = \begin{pmatrix} \sin D_0 & \cos D_0 & 0 & 0 \\ 0 & 0 & \sinh(B_0) & \cosh(B_0) \\ \sqrt{-1-k_0}\cos D_0 & -\sqrt{-1-k_0}\sin D_0 & 0 & 0 \\ 0 & 0 & \sqrt{1-k_0}\cosh(B_0) & \sqrt{1-k_0}\sinh(B_0) \end{pmatrix}, \quad (33)$$

and $(\lambda_1 = k_1, \lambda_2 = k_2)$,

$$\sigma_1^{(j)} = \frac{-r_2^{(j)}\sinh(B_j) - r_3^{(j)}\cosh(B_j)}{\sin D_j - r_1^{(j)}\cos D_j},$$

$$\sigma_2^{(j)} = \sqrt{-1-k_j}\frac{\cos D_j + r_1^{(j)}\sin D_j}{\sin D_j - r_1^{(j)}\cos D_j},$$

$$\sigma_3^{(j)} = \sqrt{1-k_j}\frac{-r_2^{(j)}\cosh B_j - r_3^{(j)}\sinh(B_j)}{\sin D_j - r_1^{(j)}\cos D_j}, \quad j = 1,2,$$

where

$$B_i = \sqrt{1-k_i}x - (2k_i+1)\sqrt{1-k_i}t, \quad D_i = \sqrt{-1-k_i}x + (2k_i-1)\sqrt{-1-k_i}t, \quad i = 0,1,2.$$

An explicit solution of the generalized Hirota-Satsuma coupled Eq. (1) is written as

$$\bar{u} = 1 + 2\left(\frac{(k_1+k_2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}\right)_x,$$

$$\bar{v} = -2\frac{k_2-k_1}{\sigma_1^{(1)}-\sigma_1^{(2)}} - 2\frac{(k_1+k_2)(\sigma_2^{(2)}-\sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}},$$

$$\bar{w} = 2\frac{(k_1-k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)}-\sigma_1^{(2)}} + 2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)}-\sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}. \quad (34)$$

(3) Choose a trivial solution $u=1, v=0, w=1$ of the generalized Hirota-Satsuma coupled Eq. (1). Then Eqs. (3) and (4) are reduced to

$$\begin{cases} \phi_{1,xx} = (1+\lambda)\phi_1, & \phi_{1,t} = (2\lambda-1)\phi_{1,x}, & \phi_3 = \phi_{1,x}, \\ \phi_{2,xx} = \phi_1 + (1-\lambda)\phi_2, & \phi_{2,t} = 2\phi_{1,x} - (2\lambda+1)\phi_{2,x}, & \phi_4 = \phi_{2,x}. \end{cases} \quad (35)$$

(i) Let $\lambda = k_0 (|k_0| < 1)$. Equation (35) has a fundamental matrix of solutions

$$\Phi = \begin{pmatrix} 2k_0\exp(A_0) & 2k_0\exp(-A_0) & 0 & 0 \\ \exp(A_0) & \exp(-A_0) & \exp(B_0) & \exp(-B_0) \\ 2k_0\sqrt{1+k_0}\exp(A_0) & -2k_0\sqrt{1+k_0}\exp(-A_0) & 0 & 0 \\ \sqrt{1+k_0}\exp(A_0) & -\sqrt{1+k_0}\exp(-A_0) & \sqrt{1-k_0}\exp(B_0) & -\sqrt{1-k_0}\exp(-B_0) \end{pmatrix}, \quad (36)$$

which together with Eq. (11) leads to $(\lambda_1 = k_1, \lambda_2 = k_2)$,

$$\sigma_1^{(j)} = \frac{1}{2k_j}\left(1 - \frac{r_2^{(j)}\exp(B_j) + r_3^{(j)}\exp(-B_j)}{\exp(A_j) - r_1^{(j)}\exp(-A_j)}\right), \quad \sigma_2^{(j)} = \sqrt{1+k_j}\frac{\exp(A_j) + r_1^{(j)}\exp(-A_j)}{\exp(A_j) - r_1^{(j)}\exp(-A_j)},$$

$$\sigma_3^{(j)} = \frac{1}{2k_j}\frac{\sqrt{1+k_j}[\exp(A_j)+r_1^{(j)}\exp(-A_j)]+\sqrt{1-k_j}[-r_2^{(j)}\exp(B_j)+r_3^{(j)}\exp(-B_j)]}{\exp(A_j)-r_1^{(j)}\exp(-A_j)}, \quad j=1,2,$$

where

$$A_i=\sqrt{1+k_i}x+(2k_i-1)\sqrt{1+k_i}t, \quad B_i=\sqrt{1-k_i}x-(2k_i+1)\sqrt{1-k_i}t, \quad i=0,1,2.$$

Therefore, we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1),

$$\bar{u}=1+2\left(\frac{(k_1+k_2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}\right)_x,$$

$$\bar{v}=-2\frac{k_2-k_1}{\sigma_1^{(1)}-\sigma_1^{(2)}}-2\frac{(k_1+k_2)(\sigma_2^{(2)}-\sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}},$$

$$\bar{w}=1+2\frac{(k_1-k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)}-\sigma_1^{(2)}}+2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)}-\sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}. \tag{37}$$

(ii) Let $\lambda=k_0(k_0>1)$. A fundamental matrix of solutions of Eq. (35) is written as

$$\Phi=\begin{pmatrix} 2k_0\sinh(A_0) & 2k_0\cosh(A_0) & 0 & 0 \\ \sinh(A_0) & \cosh(A_0) & \sin C_0 & \cos C_0 \\ 2k_0\sqrt{1+k_0}\cosh(A_0) & 2k_0\sqrt{1+k_0}\sinh(A_0) & 0 & 0 \\ \sqrt{1+k_0}\cosh(A_0) & \sqrt{1+k_0}\sinh(A_0) & \sqrt{k_0-1}\cos C_0 & -\sqrt{k_0-1}\sin C_0 \end{pmatrix}, \tag{38}$$

and we have $(\lambda_1=k_1,\lambda_2=k_2)$

$$\sigma_1^{(j)}=\frac{1}{2k_j}\left(1-\frac{r_2^{(j)}\sin C_j+r_3^{(j)}\cos C_j}{\sinh(A_j)-r_1^{(j)}\cosh(A_j)}\right), \quad \sigma_2^{(j)}=\sqrt{1+k_j}\frac{\cosh(A_j)-r_1^{(j)}\sinh(A_j)}{\sinh(A_j)-r_1^{(j)}\cosh(A_j)},$$

$$\sigma_3^{(j)}=\frac{1}{2k_j}\frac{\sqrt{1+k_j}[\cosh(A_j)-r_1^{(j)}\sinh(A_j)]+\sqrt{k_j-1}(-r_2^{(j)}\cos C_j+r_3^{(j)}\sin C_j)}{\sinh(A_j)-r_1^{(j)}\cosh(A_j)}, \quad j=1,2,\text{where}$$

$$A_i=\sqrt{1+k_i}x+(2k_i-1)\sqrt{1+k_i}t, \quad C_i=\sqrt{k_i-1}x-(2k_i+1)\sqrt{k_i-1}t, \quad i=0,1,2.$$

By means of the Darboux transformation, we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1)

$$\bar{u}=1+2\left(\frac{(k_1+k_2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}\right)_x,$$

$$\bar{v}=-2\frac{k_2-k_1}{\sigma_1^{(1)}-\sigma_1^{(2)}}-2\frac{(k_1+k_2)(\sigma_2^{(2)}-\sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}},$$

$$\bar{w}=1+2\frac{(k_1-k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)}-\sigma_1^{(2)}}+2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)}-\sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}}. \tag{39}$$

(iii) Let $\lambda=k_0(k_0<-1)$. Equation (35) has a fundamental matrix of solutions

$$\Phi=\begin{pmatrix} 2k_0\sin D_0 & 2k_0\cos D_0 & 0 & 0 \\ \sin D_0 & \cos D_0 & \sinh(B_0) & \cosh(B_0) \\ 2k_0\sqrt{-1-k_0}\cos D_0 & -2k_0\sqrt{-1-k_0}\sin D_0 & 0 & 0 \\ \sqrt{-1-k_0}\cos D_0 & -\sqrt{-1-k_0}\sin D_0 & \sqrt{1-k_0}\cosh(B_0) & \sqrt{1-k_0}\sinh(B_0) \end{pmatrix}, \tag{40}$$

and $(\lambda_1=k_1,\lambda_2=k_2)$

$$\sigma_1^{(j)}=\frac{1}{2k_j}\left(1-\frac{r_2^{(j)}\sinh(B_j)+r_3^{(j)}\cosh(B_j)}{\sin D_j-r_1^{(j)}\cos D_j}\right), \quad \sigma_2^{(j)}=\sqrt{-1-k_j}\frac{\cos D_j+r_1^{(j)}\sin D_j}{\sin D_j-r_1^{(j)}\cos D_j},$$

$$\sigma_3^{(j)} = \frac{1}{2k_j}\frac{\sqrt{-1-k_j}(\cos D_j + r_1^{(j)}\sin D_j) - \sqrt{1-k_j}[r_2^{(j)}\cosh B_j + r_3^{(j)}\sinh (B_j)]}{\sin D_j - r_1^{(j)}\cos D_j}, \quad j=1,2,$$

where

$$B_i = \sqrt{1-k_i}x - (2k_i+1)\sqrt{1-k_i}t, \quad D_i = \sqrt{-1-k_i}x + (2k_i - 1)\sqrt{-1-k_i}t, \quad i=0,1,2.$$

Therefore, we get an explicit solution of the generalized Hirota-Satsuma coupled Eq. (1)

$$\bar{u} = 1 + 2\left(\frac{(k_1+k_2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}\right)_x,$$

$$\bar{v} = -2\frac{k_2-k_1}{\sigma_1^{(1)}-\sigma_1^{(2)}} - 2\frac{(k_1+k_2)(\sigma_2^{(2)}-\sigma_2^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}},$$

$$\bar{w} = 1 + 2\frac{(k_1-k_2)\sigma_1^{(1)}\sigma_1^{(2)}}{\sigma_1^{(1)}-\sigma_1^{(2)}} + 2\frac{(k_1+k_2)(\sigma_1^{(1)}\sigma_3^{(2)} - \sigma_1^{(2)}\sigma_3^{(1)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}. \tag{41}$$

B. Explicit solutions of the complex coupled KdV Eq. (2)

It is obvious that $(u,v)=(0,0)$ is a trivial solution of the complex coupled KdV Eq. (2). In this case, Eqs. (3) and (4) are reduced to

$$\begin{cases} \phi_{1,xx} = \lambda\phi_1, & \phi_{1,t} = 2\lambda\phi_{1,x}, & \phi_3 = \phi_{1,x}, \\ \phi_{2,xx} = -\lambda\phi_2, & \phi_{2,t} = -2\lambda\phi_{2,x}, & \phi_4 = \phi_{2,x}, \end{cases} \tag{42}$$

with $\lambda=(i\xi_0)^2$. Then we have a fundamental matrix of solutions of Eq. (42),

$$\Phi = \begin{pmatrix} \exp(-B_0^*) & \exp(B_0^*) & 0 & 0 \\ 0 & 0 & \exp(A_0) & \exp(-A_0) \\ i\xi_0\exp(-B_0^*) & -i\xi_0\exp(B_0^*) & 0 & 0 \\ 0 & 0 & \xi_0\exp(A_0) & -\xi_0\exp(-A_0) \end{pmatrix}, \tag{43}$$

which together with Eq. (11) leads to

$$\sigma_1^{(1)} = \frac{-r_2^{(1)}\exp(A_1) - r_3^{(1)}\exp(-A_1)}{\exp(-B_1^*) - r_1^{(1)}\exp(B_1^*)},$$

$$\sigma_1^{(2)} = \frac{-r_2^{(2)}\exp(-B_1) - r_3^{(2)}\exp(B_1)}{\exp(A_1^*) - r_1^{(2)}\exp(-A_1^*)},$$

$$\sigma_2^{(1)} = i\xi_1\frac{\exp(-B_1^*) + r_1^{(1)}\exp(B_1^*)}{\exp(-B_1^*) - r_1^{(1)}\exp(B_1^*)},$$

$$\sigma_2^{(2)} = \xi_1^*\frac{\exp(A_1^*) + r_1^{(2)}\exp(-A_1^*)}{\exp(A_1^*) - r_1^{(2)}\exp(-A_1^*)},$$

$$\sigma_3^{(1)} = -\xi_1\frac{r_2^{(1)}\exp(A_1) - r_3^{(1)}\exp(-A_1)}{\exp(-B_1^*) - r_1^{(1)}\exp(B_1^*)},$$

$$\sigma_3^{(2)} = i\xi_1^*\frac{r_2^{(2)}\exp(-B_1) - r_3^{(2)}\exp(B_1)}{\exp(A_1^*) - r_1^{(2)}\exp(-A_1^*)},$$

where

$$A_j = \xi_j x + 2\xi_j^3 t, \quad B_j = i\xi_j^* x - 2i\xi_j^{*3}t (j=0,1), \quad \lambda_1 = (i\xi_1)^2, \quad \lambda_2 = -\lambda_1^*(\xi_2 = -i\xi_1^*).$$

Utilizing the Darboux transformation and $\frac{r_3^{(1)}}{r_2^{(1)}} = -r_1^{(2)*}$, $r_2^{(1)}r_2^{(2)*}=1$, $\frac{r_1^{(1)}}{r_2^{(1)}} = -r_3^{(2)*}$, we get an explicit solution of the complex coupled KdV Eq. (2),

$$\bar{u} = 2\left(\frac{(\xi_1^{*2}-\xi_1^2)(\sigma_1^{(1)}-\sigma_1^{(2)})}{\sigma_1^{(2)}\sigma_2^{(1)} - \sigma_1^{(1)}\sigma_2^{(2)} + \sigma_3^{(1)} - \sigma_3^{(2)}}\right)_x = 4\left(\frac{\kappa_1\kappa_2[r_2^{(1)}|r_1^{(2)}|\cosh\Delta_1 - r_2^{(2)}|r_1^{(1)}|\cosh\Delta_2 - r_2^{(1)}\,\mathrm{Re}\,\Delta_3 + r_2^{(2)}\,\mathrm{Re}\,\Delta_4]}{\kappa_2 r_2^{(1)}|r_1^{(2)}|\sinh\Delta_1 + \kappa_1 r_2^{(2)}|r_1^{(1)}|\sinh\Delta_2 + \kappa_1 r_2^{(1)}\,\mathrm{Im}\,\Delta_3 + \kappa_2 r_2^{(2)}\,\mathrm{Im}\,\Delta_4}\right)_x, \tag{44}$$

$$\begin{aligned}\bar{v} &= -2(\xi_1^{*2}+\xi_1^2)\frac{1}{\sigma_1^{(1)}-\sigma_1^{(2)}} - 2(\xi_1^{*2}-\xi_1^2)\frac{\sigma_2^{(2)}-\sigma_2^{(1)}}{\sigma_1^{(2)}\sigma_2^{(1)}-\sigma_1^{(1)}\sigma_2^{(2)}+\sigma_3^{(1)}-\sigma_3^{(2)}} \\ &= 2[(\kappa_1)^2-(\kappa_2)^2]\frac{\exp(\Delta_5)+r_1^{(1)}r_1^{(2)}\exp(-\Delta_5)-r_1^{(1)}\exp(\Delta_6)-r_1^{(2)}\exp(-\Delta_6)}{r_2^{(1)}|r_1^{(2)}|\cosh\Delta_1 - r_2^{(2)}|r_1^{(1)}|\cosh\Delta_2 - r_2^{(1)}\,\mathrm{Re}\,\Delta_3 + r_2^{(2)}\,\mathrm{Re}\,\Delta_4} \\ &\quad + 2\kappa_1\kappa_2\frac{(\xi_1^*-i\xi_1)[\exp(\Delta_5)-r_1^{(1)}r_1^{(2)}\exp(-\Delta_5)]-(\xi_1^*+i\xi_1)[r_1^{(1)}\exp(\Delta_6)-r_1^{(2)}\exp(-\Delta_6)]}{\kappa_2 r_2^{(1)}|r_1^{(2)}|\sinh\Delta_1+\kappa_1 r_2^{(2)}|r_1^{(1)}|\sinh\Delta_2+\kappa_1 r_2^{(1)}\,\mathrm{Im}\,\Delta_3+\kappa_2 r_2^{(2)}\,\mathrm{Im}\,\Delta_4},\end{aligned} \tag{45}$$

where

$$\Delta_1 = \ln|r_1^{(2)}| - 2\,\mathrm{Re}\,A_1, \quad \Delta_2 = \ln|r_1^{(1)}| + 2\,\mathrm{Re}\,B_1, \quad \Delta_3 = r_1^{(2)}\exp(2i\,\mathrm{Im}\,A_1),$$

$$\Delta_4 = r_1^{(1)}\exp(-2i\,\mathrm{Im}\,B_1), \quad \Delta_5 = A_1^* - B_1^*, \quad \Delta_6 = A_1^* + B_1^*, \quad \mathrm{Re}\,\xi_1 = \kappa_1, \quad \mathrm{Im}\,\xi_1 = \kappa_2.$$

A simple calculation shows that $\bar{u}$ given by Eq. (44) is a real function. This means that $\bar{u}$, $\bar{v}$ determined by Eqs. (44) and (45) with $\bar{w}=\bar{v}^*$ (complex conjugate) is a complex solution of the generalized Hirota-Satsuma coupled KdV Eq. (1).

V. CONCLUSIONS

In the present paper, we have given an approach to construct a Darboux transformation for the generalized Hirota-Satsuma coupled KdV equation based on a gauge transformation between the corresponding 4×4 matrix spectral problems with three potentials. Here the Darboux matrix T to be suitably chosen is the key to construct the Darboux transformation, by which the Lax pair of the generalized Hirota-Satsuma coupled KdV equation is changed into another Lax pair of the same type. A Darboux transformation of the complex coupled KdV equation is obtained through the reduction technique. As applications of Darboux transformations, some explicit solutions of the generalized Hirota-Satsuma coupled KdV equation and the complex coupled KdV equation are given explicitly.

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